15.5 & 15.7 : Mass, Moments & Centers of Mass

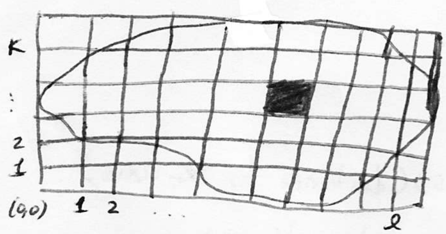
Goal: Find the center of mass of a lamina (this is, the point where if you put your finger, the lamina will balance).

Outline: I. Mass & Density
II. Moments
III. Center of Mass. (to compute center of mass, we need to know how to compute mass & moments).

I. Mass & Density

When we have a lamina D  with uniform density ρ, we know $\text{Mass} = \rho \cdot A$ where A is the area of the lamina.

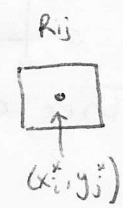
→ If the density is not uniform (say density is $\rho(x,y)$):



→ Compute the mass at each square & let # squares ⟶ ∞.

Mass of square R_{ij} is appx.

$$\rho(x_i^*, y_j^*)\,\Delta A.$$



$$\text{Mass of } D \approx \sum_{j=1}^{k} \sum_{i=1}^{\ell} \rho(x_i^*, y_j^*)\,\Delta A.$$

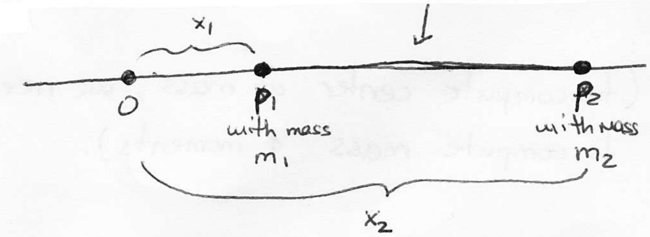
letting the # of squares ⟶ ∞, $\text{Mass } D = \lim_{\substack{k\to\infty \\ \ell\to\infty}} \sum_{j=1}^{k} \sum_{i=1}^{\ell} \rho(x_i^*, y_j^*)\,\Delta A$

$$\boxed{\text{Mass} = \iint_D \rho(x,y)\, dA.}$$

II. Moments

Recall, the moment of mass of a point in $\mathbb{R}^2$ about the x-axis (resp. y-axis) is the mass of the point times the distance from the point to the x-axis (resp. y-axis).

In $\mathbb{R}$



$m_1 x_1$ is the moment of P_1

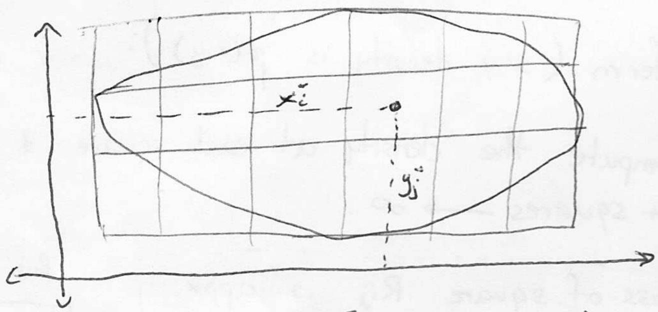
$m_2 x_2$ is the moment of the P_2

$\bar{x}$ = center mass = $\dfrac{m_1 x_1 + m_2 x_2}{\cancel{\text{total}}\ \text{mass}}$

So $m_2 x_2$ measures the tendency of the person P_2 to be on the ground.

In $\mathbb{R}^2$

We define the moment of a lamina D about the x-axis in a similar way.



First, take a square  & compute (mass)(distance to x-axis).

(x_i^*, y_j^*)

$$\approx (\rho(x_i^*, y_j^*)\Delta A)(y_j^*).$$

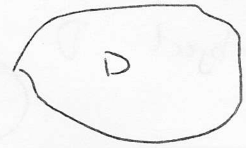
Then, sum over all squares to get

$$\boxed{M_x = \text{moment of } D \text{ about the x-axis} = \iint_D y\,\rho(x,y)\,dA.}$$

Similarly

$$\boxed{M_y = \text{moment of } D \text{ about the y-axis} = \iint_D x\,\rho(x,y)\,dA.}$$

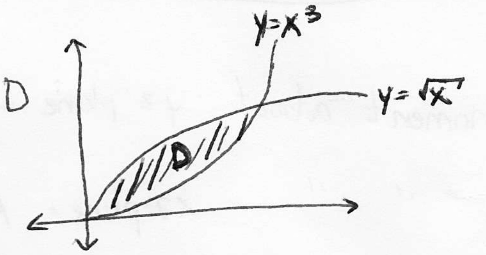
III. Center of Mass.

Thm: The center of mass of a lamina  with density $\rho(x,y)$

is $(\bar{x}, \bar{y})$ where

$$\boxed{\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}}$$

Example:

Find the center of mass of the lamina D 

with density $\rho(x,y) = 4x$

step 1) Find the mass of D

$$\text{mass of } D = \iint_D \rho(x,y)\,dA = \iint_D 4x\,dA$$

$$= \int_0^1 \int_{x^3}^{\sqrt{x}} (4x)\,dy\,dx = \int_0^1 4xy\Big]_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 4(x^{3/2} - x^4)\,dx = 4x^{5/2}(2/5) - \frac{4x^5}{5}\Big]_0^1 = \frac{8}{5} - \frac{4}{5} = \frac{4}{5}$$

step 2) Find M_x, M_y

$$M_y = \iint_D y\,(\rho(x,y))\,dA = \iint_D 4xy\,dA$$

$$= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy\,dy\,dx = \int_0^1 2xy^2\Big]_{x^3}^{\sqrt{x}} dx = \int_0^1 (2x^2 - 2x^7)\,dx = \frac{2x^3}{3} - \frac{2x^8}{8}\Big]_0^1$$

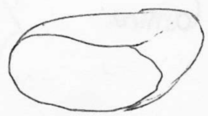
$$= \frac{2}{3} - \frac{1}{4} = 5/12$$

$$M_x = \iint_D x\,\rho(x,y)\,dA = 10/21$$

step 3) $\bar{x} = \dfrac{M_y}{m} = \dfrac{5/12}{4/5} = \dfrac{25}{48}$ $\qquad \bar{y} = \dfrac{M_x}{m} = \dfrac{10/21}{4/5} = 50/84 = 25/42$

$$\boxed{\text{Center of mass} = (25/48,\ 25/42)}$$

$\boxed{\text{In } \mathbb{R}^3}$, everything extends nicely.

We have a solid object D  with density $\rho(x,y,z)$.

$$\boxed{\text{Mass } D = \iiint_D \rho(x,y,z)\, dV.}$$

→ moment about y-z plane $= M_{yz} = \iiint_D x\, \rho(x,y,z)\, dV$

→ " " xz-plane $= M_{xz} = \iiint_D y\, \rho(x,y,z)\, dV$

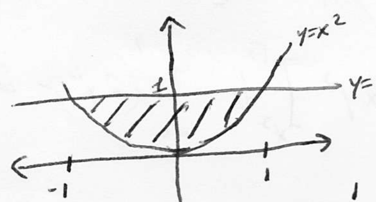
→ " " xy plane $= M_{xy} = \iiint_D z\, \rho(x,y,z)\, dV$

Center of mass $= (\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{\bar{x} = \frac{M_{y,z}}{m}}, \quad \boxed{\bar{y} = \frac{M_{xz}}{m}}, \quad \boxed{\bar{z} = \frac{M_{xy}}{m}}$$

Example: Find the center of mass of the solid bounded by $y = x^2$, $y = 1$, $z = 0$, $z = 4$, with density $\rho(x,y,z) = 4$.

Step 0) Draw region:



This is the base region in the xy plane. Then lift this region from $z=0$ to $z=4$.

Step 1) mass of $D = \iiint_D \rho(x,y,z)\, dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{4} 4\, dz\, dy\, dx$

$$= \int_{-1}^{1}\int_{x^2}^{1} 4z \Big]_0^4 dy\, dx = \int_{-1}^{1}\int_{x^2}^{1} 16\, dy\, dz = \int_{-1}^{1} 16y \Big]_{x^2}^{1}$$

$$= \int_{-1}^{1} (16 - 16x^2)\, dx = 16x - \frac{16x^3}{3} \Big]_{-1}^{1}$$

$$= 16 - \frac{16}{3} - \left(-16 + \frac{16}{3}\right) = 32 - \frac{2(16)}{3} \overset{?}{=} \frac{64}{3}$$

Step 2) $M_{yz} = \iiint\limits_D x\,\rho(x,y,z)\,dV$

$$= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{4} 4x\,dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} 16x\,dy\,dx$$

$$= \int_{-1}^{1} 16x - 16x^3\,dx = \frac{16x^2}{2} - \frac{16x^4}{4}\Big]_{-1}^{1}$$

$$= \frac{16}{2} - \frac{16}{4} - \left(\frac{16}{2} - \frac{16}{4}\right) = 0$$

$M_{xz} = \iiint\limits_D y\,\rho(x,y,z)\,dV$

$$= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{4} 4y\,dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} 16y\,dy\,dx$$

$$= \int_{-1}^{1} 8y^2\Big]_{x^2}^{1} dx = \int_{-1}^{1} 8 - 8x^2\,dx = 8x - \frac{8x^3}{3}\Big]_{-1}^{1}$$

$$= 8 - 8/3 - (-8 + 8/3) = 16 - \frac{2(8)}{3} = \frac{32}{3}$$

$$M_{xy} = \iiint\limits_D z\,\rho(x,y,z)\,dz\,dy\,dx = \iiint\limits_D 4z\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} 2z^2\Big]_{0}^{4} dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} 32\,dy\,dx = \int_{-1}^{1} 32 - 32x^2\,dx$$

$$= 32x - \frac{32x^3}{3}\Big]_{-1}^{1} = 32 - \frac{32}{3} - \left(-32 + \frac{32}{3}\right) = 64 - \frac{2(32)}{3} = \frac{128}{3}$$

$\boxed{\bar{x} = \frac{0}{m} = 0}$, $\boxed{\bar{y} = \frac{32}{3}\Big/\frac{64}{3} = \frac{1}{2}}$, $\boxed{\bar{z} = \left(64 - \frac{64}{3}\right)\Big/\frac{64}{3} = 3 - 1 = 2}$

Applications:

1) Why do car manufacturers make sport cars so close to the ground? Where is the center of mass?

Why do SUV tend to flip more often than cars?

2) When constructing ANY object, one has to be aware of the position of the center of mass. For example:

a. What happens if the battery of a cell phone is located at the top of the phone (instead of a long bar that goes from top to bottom)?

b. What happens if the center of mass of a plane is located in the front part of the plane?

3) Gymnasts, high jumpers and other sport people have to be aware of where the center of mass of their body is, since this may afect their landings, jumpings, etc.

For more applications, google "center of mass applications".